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The benefits of education and of useful knowledge, generally diffused through a community, are essential to the preservation of a free government.

Sam Houston

Cultivated mind is the guardian genius of Democracy, and while guided and controlled by virtue, the noblest attribute of man. It is the only dictator that freemen acknowledge, and the only security which freemen desire.

Mirabeau B. Lamar

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THE TEXAS MATHEMATICS TEACHERS' BULLETIN

Volume XVI, Number 2

Edited by

P. M. BATCHELDER
Associate Professor of Pure Mathematics

and

MARY E. DECHERD
Adjunct Professor of Pure Mathematics

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CONTENTS

The Calendar	P. M. Batchelder	5
Some Mistakes I Made in Teaching High-School Mathematics	M. M. Abernathy	17
"Lessons in Mathematics"	C. E. Castañeda	22
The Meanings of Numbers	S. B. Red	26
Interpolation	R. J. Gonzalez	32

THE CALENDAR

BY P. M. BATCHELDER
The University of Texas

For the accurate measurement of time mankind is dependent upon regularly recurring or periodic phenomena, such as the oscillation of a pendulum or spring for short intervals, and the movements of the heavenly bodies for longer periods. Accuracy in the measurement of short intervals has been attained only in modern times, by the invention of clocks, watches, and chronometers; the ancients had to rely on such crude devices as hour glasses and sun dials. All civilized peoples, however, have felt the need of some scheme for dividing up long periods of time, and have devised various types of calendar for this purpose, based on astronomical phenomena.

The regular alternation of light and darkness, on which the routine of our daily lives depends so closely, furnishes the simplest and most natural unit of time, the day, which may be defined as the interval between two successive passages of the sun across the meridian. The apparent motion of the sun around the earth once a day is due to the rotation of the earth on its axis; the length of the day is not exactly equal to the period of the rotation, however, but is about four minutes longer, because the earth at the same time is moving in its orbit around the sun in the same direction, so that it has to turn through a little more than one complete revolution to bring the sun back to the meridian. The day as thus defined varies slightly in length, partly because the motion of the earth around the sun is not perfectly uniform, and partly because the axis of the earth's rotation is not perpendicular to the plane of its orbit. To avoid the inconvenience of a variable day, modern nations use the "mean solar day," which is the average value for the whole year of the day defined above. Our smaller units of time (hours, minutes, and seconds) are subdivisions of this mean solar day.

Next to the day, the natural rhythm which impresses man most profoundly, at least in the temperate regions of the earth where the highest civilizations have developed, is the sequence of summer and winter, of seedtime and harvest, on which the production of most of our food depends. The period of this rhythm is called a year (more precisely, a "tropical year"), and is the interval between two successive passages of the sun across the celestial equator from south to north. This is approximately the period of the earth's revolution around the sun (called the "siderial year"), but is about twenty minutes shorter than the latter, on account of a slow change in the position of the points where the sun's path crosses the equator, known as the "precession of the equinoxes."

Since the year is very much longer than the day, most peoples have found it convenient to use a time unit of intermediate length. The waxing and waning of the moon, while they have very little influence on human affairs, are so conspicuous that they have been almost universally employed for this purpose. The interval between two successive new moons is called a lunar month; it is about two days and five hours longer than the period of revolution of the moon around the earth, the difference being due to the motion of the earth in its orbit around the sun.

Another intermediate unit, which plays a prominent part in our own calendar, but which has been of much less general use than the month, is the week of seven days. The ultimate origin of the week is uncertain; probably it arose from some mystic significance attached to seven as a sacred number, though it is possible that it merely represents one-fourth of a month. It seems to have been borrowed from the Babylonians by the ancient Jews, from whom it passed to the Christians and Mohammedans, who have spread it over a large part of the modern civilized world.

Unfortunately for those who have tried to base a calendar on the day, month, and year, the lengths of these periods are entirely incommensurable; the average duration of the lunar month is 29 days, 12 hours, 44 minutes, and 3 seconds, while the year consists of 365 days, 5 hours, 48 minutes,

and 46 seconds. The attempts to reconcile these periods have led to endless difficulties, and a wholly satisfactory adjustment has not even yet been worked out.

One method of avoiding these difficulties is frankly to abandon the use of one of the units; thus Mohammed, in establishing a calendar for his followers, based it on the moon and ignored the seasons. The Mohammedan "year" consists of twelve lunar months of 29 or 30 days each, totaling 354 or 355 days; consequently a month which comes in the middle of winter in one year will after the lapse of 17 years come in the middle of summer. Our own calendar has abandoned the use of the moon; the periods which we call "months" are merely conventional subdivisions of the year, and have no connection with the phases of the moon. Modern astronomers use an interesting system of reckoning in which both months and years are dispensed with; beginning with January 1, 4713 B.C., the days (running from noon to noon) are numbered consecutively and called "Julian days." The day which begins at noon on October 1, 1932, is J. D. 2,426,982. This system leads to very large numbers, but for scientific purposes it has many merits.

The ancient Greeks made use of months of 29 or 30 days, but, unlike the Mohammedans, they sought to harmonize their calendar with the solar year by inserting an extra or intercalary month at suitable intervals. About 432 B.C. Meton discovered that 19 years are almost exactly equal to 235 lunar months, a fact which enabled the intercalations to be made in a systematic fashion. The 19-year period, after which the phases of the moon recur on the same days of the month, is called a Metonic cycle from his name.

The Romans of the republican period had a similar calendar, but the decision as to when an intercalary month should be inserted was left to certain religious officials, who administered their duty in a highly inefficient manner, and were even accused of manipulating the calendar for political purposes. The resulting confusion and dislocation of the calendar were so great that in 46 B.C. Julius Caesar decided to make a thorough-going reform, and carried out his purpose in such a masterly fashion that only one slight

modification of his scheme has been found necessary in the 2,000 years since. Following the advice of the Alexandrian astronomer Sosigenes, Caesar adopted 365¼ days as the length of the year; to reduce intercalations to a minimum, he decreed that every fourth year should contain 366 days, and all the others 365. The use of lunar months was abandoned, each year being divided into 12 conventional months of prescribed length. The revised calendar went into effect in 45 B.C., following some special adjustments in the year 46. The extra day in leap years was provided for by counting the 24th of February twice, the two days being reckoned legally as one day. The 24th of February according to the Roman practice was called the sixth (Latin *sextus*) day before the Calends of March, and from its being counted twice (*bis*), leap years came to be known as "bissextile" years. Later the extra day was transferred to the end of the month.

This Julian calendar was not officially imposed on the whole empire, but the example of the capital was quickly followed throughout most of the Roman world. It was a vast improvement over those which it superseded, and for many centuries its lack of complete adjustment to the seasons attracted no serious notice. The year of 365¼ days, however, was 11 minutes and 14 seconds too long, and the accumulated discrepancy amounted to a full day every 128 years.

It was in connection with the date of Easter that the discrepancy first caused trouble. In the early days of Christianity there was disagreement as to when this festival should be celebrated; some churches followed the date of the Jewish Passover, the 14th of the month Nisan, which might fall on any day of the week; others celebrated the following Sunday, etc. The question was discussed at length in the Council of Nicaea, 325 A.D., and the rule was ultimately adopted (whether at this time or later is not quite certain) that Easter should be the Sunday immediately following the full moon which occurs on or first after the vernal equinox, the date of which was taken as March 21.

The error in the length of the Julian year caused the calendar date of the vernal equinox to recede gradually toward the beginning of the month; by the sixteenth century it had reached March 11, and serious disputes arose as to whether March 21 or the actual date of the vernal equinox should be used in determining the date of Easter. A further difficulty intervened from the methods used in computing the position of the moon. Definite rules had been adopted by the Church for determining the dates of new and full moons, based on the two assumptions that the year contains exactly 365¼ days and that 19 years are exactly equal to 235 lunar months, whereas both of these relations are only approximately true. The result was that the fictitious moon of the ecclesiastical computations gradually drew away from the real moon, and in the sixteenth century they were four days apart.

Numerous proposals were made for reforming the calendar, and in 1577 Pope Gregory XIII appointed a commission presided over by the German mathematician Christopher Schlüssel, or Clavius, as he is better known, to consider the matter. Their recommendations were adopted, and in 1582 Gregory promulgated the revised calendar, which provided that in the future the years divisible by 100 should not be leap years unless they are divisible by 400; thus 1700, 1800, and 1900 were ordinary years, but 2000 will be a leap year. To bring the vernal equinox back to March 21, Gregory decreed that ten days should be omitted from the year 1582, October 4 being followed immediately by October 15. At the same time the position of the ecclesiastical moon was corrected, and the rules of computation revised so as to insure greater accuracy in the future.

Gregory's reform was accepted promptly in Roman Catholic countries, but only slowly by the Protestant states. In England and her colonies the change was made by Act of Parliament in 1752, 11 days being omitted from September of that year. The change was widely misunderstood, and riots occurred at a number of places, in which several people were killed; the populace felt that they were being robbed of something tangible, and demanded that the 11

days be given back to them. Russia and several other eastern European countries retained the Julian calendar until after the World War. In changing to the Gregorian they had to drop 13 days, since the discrepancy increased by one day in each of the years 1700, 1800, and 1900, which were leap years in the Julian reckoning but not in the Gregorian. The Gregorian calendar has also been adopted during the last 60 years by Japan, China, and Turkey as part of their adaptation to modern western civilization.

The Gregorian calendar is an improvement over the Julian, but modern historians and chronologers are agreed that it was a mistake to make the change retroactive: the advantages of restoring the vernal equinox to March 21 were far from compensating for the endless confusion which resulted from dropping days from the calendar and upsetting the dates of historical events over a period of more than a thousand years.

The average length of the Gregorian year is 365 days, 5 hours, 49 minutes, and 12 seconds, which is still 26 seconds longer than the tropical year, a discrepancy which will amount to a day in about 3,300 years. If Gregory had omitted leap year once in 128 years, instead of three times in 400 years, he would have obtained a much more accurate calendar, one in which the error would not amount to a day for more than 100,000 years. Sir John Herschel proposed that the Gregorian calendar be modified by counting years divisible by 4,000 as ordinary years instead of leap years, but the change required is so far in the future that no action is likely to be taken for many centuries. In 1923 the Greek Orthodox Church adopted the rule that century years shall be leap years if when the number of the year is divided by 9 the remainder is 2 or 6, but otherwise ordinary years. Thus two century years in each nine are to be leap years, instead of one in four as in the Gregorian calendar: the net effect is to shorten the calendar by one day in 3,600 years, a more accurate correction than that proposed by Herschel. The error in the average length of the year is reduced to less than three seconds. The first departure of

this Orthodox calendar from the Gregorian will be in the year 2800.

In fixing the dates of events, it is necessary in some way to distinguish one year from another. One method of doing this is to give the name of some official who held office during the year; thus in ancient Athens one of the nine archons chosen each year was called the "eponymous archon," and his name identified the year; similarly in Rome the names of the consuls were used. A common practice in kingdoms and empires was to use the regnal years of the monarch: "And it came to pass in the fifth year of king Rehoboam, that Shishak king of Egypt came up against Jerusalem" (1 Kings 14:25).

Such methods are adequate for a generation or two, but become exceedingly cumbersome and confusing when applied to long periods of time; the only satisfactory scheme is to reckon dates from some fixed epoch. When the Alexandrian Greeks began a systematic study of the history of their race, about 300 B.C., they realized this need, and devised the well-known system of Olympiads to meet it. The Olympian games were celebrated in the middle of summer every fourth year, and lists of the victors in the various contests were carefully preserved. The Alexandrians succeeded in reconstructing lists of the victors back to 776 B.C., which date they accordingly took as the beginning of the first Olympiad. The early lists were probably largely legendary, so that the date has no real historical significance, but for chronological purposes that does not matter. Any year (running from July 1 to July 1) could be denoted by the number of the Olympiad and the number of the year in that Olympiad; thus the battle of Plataea was fought in September, Ol. 75, 2 (479 B.C). The Alexandrians also made some use of the "era of Nabonassar," which dated from 747 B.C. Nabonassar was king of Babylon from 747 to 734 B.C. We have little authentic knowledge about his reign; later writers assert that he established a new era and revised the Babylonian calendar, but there is no evidence that the Babylonians themselves ever used the era.

A century or so later various Roman antiquarians sought to determine from their early legends the date of the founding of Rome; the results of their computations did not agree, but ultimately the year 753 B.C. was fixed upon, and this became an epoch for reckoning dates. Thus Octavian received the title Augustus and formally established the imperial government in the year 727 A. U. C. (*ab urbe condita*, from the founding of the city, or *anno urbis conditae*, in the year of the founding of the city), i.e., 27 B.C.

The introduction of the Christian era, which we use, is attributed to Dionysius Exiguus, a monk who lived at Rome in the first half of the sixth century. In determining the date of the birth of Christ he seems to have followed a widespread tradition that Christ was born in the 28th year of the reign of Augustus. Whether this date is correct we are unable to decide; the indications in the Gospels are conflicting, since Matthew associates his birth with King Herod, who died in 4 B.C., and Luke with a census which apparently was that taken in 6 A.D. The use of the Christian era spread only gradually; in England it did not become general until the end of the eighth century.

The year immediately preceding the first year of the Christian era is denoted by 1 B.C., the one before that by 2 B.C., etc. From a mathematical standpoint it would be more logical to have a year 0 between 1 A.D. and 1 B.C., and in fact astronomers sometimes follow this scheme, with the result that their dates B.C. differ by unity from the usual ones. The absence of a year 0 is sometimes overlooked, for example in certain elementary algebra books which use years A.D. and B.C. to illustrate positive and negative numbers. In finding the interval between a date B.C. and one A.D. it is necessary to subtract 1 from the sum of the dates; thus from May 1, 10 B.C. to May 1, 10 A.D. is 19 years, not 20. The two thousandth anniversary of Virgil's birth, which occurred in 70 B.C., was celebrated in Italy in 1930, instead of 1931, the correct date; the error was not discovered until the preparations for the celebration were too far advanced for it to be postponed a year.

Other eras besides the Christian are in use. The Mohammedans number their lunar years from the Hegira, or flight of Mohammed from Mecca to Medina, in 622 A.D. The Jews use for religious purposes a calendar in which the years are counted from 3761 B.C., which is taken as the date of the creation of the world. Ancient eras are employed in India, China, and Japan. Many other eras have had a limited vogue, such as that of the French Revolutionists, which dated from September 22, 1792; this was abolished by Napoleon in 1805.

In recent years there has been much discussion of reforming the division of the year into months and weeks. Our present system of months is inherited from the Romans. When Julius Caesar made his reform in 46 B.C. he seems to have divided the year into 12 months of 30 or 31 days each, except February, which had 29 days in ordinary years and 30 in leap years. At this period the year of office of the consuls began on January 1; in early times the Roman year began on March 1, as is indicated by the names of the last four months of the year, which are derived from the Latin words for seven, eight, nine, and ten, although they are now the ninth, tenth, eleventh, and twelfth months. July and August in Caesar's time bore the names of Quintilis and Sextilis (i.e., fifth and sixth months), but in 44 B.C. the name Quintilis was changed to July in his honor, and in 8 B.C. Sextilis was changed to August in honor of the emperor Augustus. It is often asserted that Augustus took a day from February and added it to August, either to make his month as long as Caesar's, or because of a superstitious belief that even numbers were unlucky, but whether this is the true explanation of our 28-day month cannot now be ascertained.

The disadvantages of our present calendar may be summarized as follows. The number of days in a month varies from 28 to 31; the number of business days varies even more widely, on account of the irregular distribution of Sundays and holidays. The lengths of the quarters are unequal, since they contain 90 (or 91), 91, 92, and 92 days respectively. The first half-year is therefore shorter than

the second by three (or two) days. These irregularities cause much difficulty and confusion in economic and commercial affairs, in which statistical comparisons have assumed a large and steadily increasing importance. Furthermore, the fact that the week has no relation to the month or year makes the calendar change from year to year. January 1 may fall on any one of the seven days of the week, and the calendar of any given year never repeats itself earlier than the sixth year following, 11 years being the commonest interval; the calendar of 1932 will not recur until 1960. This variability makes it necessary to print new calendars every year, and to spend much time consulting them. Also the dates of periodical events cannot be definitely fixed; either some calendar date must be assigned, such as March 4, which may fall on Sunday, or such clumsy locutions as "the last Thursday in November" or "the Tuesday following the first Monday in November" must be resorted to. Finally, since in many lines of trade the amount of business transacted varies markedly with the day of the week, statistical comparisons of the same month in different years cannot be made satisfactorily.

The schemes for reform which have been suggested are too numerous to be detailed here. The simplest ones limit themselves to equalizing as far as possible the lengths of the months and quarters, without disturbing the sequence of weekdays. This would make statistical results for months and quarters more readily comparable, but would still leave the calendar variable from year to year. Most reform schemes, however, have as their goal a perpetual calendar, one which will be exactly the same for every year, apart from the extra day inserted in leap years. These fall into two main groups, one of which proposes 13 months of 28 days each, and the other 12 months, of which eight are to contain 30 days and four 31; both agree in placing the remaining day (or two days in leap year) outside the framework of weeks and months, so that every year would consist of 52 weeks, plus one (or two) days to which no weekday label is attached.

The 13-month calendar has the advantage that every month is exactly like every other month, and contains exactly four weeks. It would be very readily memorized, so that printed calendars would soon become obsolete. Its statistical advantages are so great that it has already been adopted by a considerable number of large corporations, such as the Eastman Kodak Company and Sears, Roebuck and Company, for their own bookkeeping purposes. The 12-month plan, in which each quarter consists of two 30-day months and one 31-day month, totaling exactly 13 weeks, has the advantages that the ends of the quarters coincide with the ends of months, and that a less violent departure from our present calendar is required. Both schemes have the practical difficulty that they involve an interruption of the regular sequence of weekdays; the advocates of these reforms probably underestimate the enormous inertia of religious conservatism which will oppose itself to any meddling with such a sacred institution as the week.

Another reform which is more or less independent of those just discussed has to do with fixing the date of Easter, on which depend also the dates of various other religious festivals, such as Ash Wednesday and Pentecost. The method explained above of determining the date of Easter allows it to fall on any day from March 22 to April 25 inclusive, a period of 35 days. Since Easter has an appreciable effect on trade, there would be commercial advantages in having the date wander less widely. This is more important in Europe than in America, for there Easter is celebrated as a secular holiday period, which in many countries extends over three or four days. The question is primarily a religious one, however, so action by the various churches is required. No serious opposition to stabilizing the date has appeared, but in the absence of any machinery for coöperative action progress is slow. The Church of England has led the way by bringing the matter before the British Parliament, which in 1928 passed a bill fixing Easter as the Sunday following the second Saturday in April. This change is not to go into effect until it has been adopted in other countries. It still allows Easter to

vary over seven days in our present calendar, but would give a fixed date if any perpetual calendar should be adopted.

In 1923 a committee of the League of Nations was appointed to study the question of calendar reform. They examined a large number of proposed schemes, but did not decide in favor of any one of them, taking the position that public opinion was not yet ready for an immediate reform. The formation of national committees in the various countries was recommended, to pursue the study further in coöperation with the League. Such a committee was formed in the United States in 1928, headed by the late George Eastman, president of the Eastman Kodak Company. In October, 1931, an international conference on calendar reform was held at Geneva under the auspices of the League. The delegates voted to support the stabilization of Easter, but felt that the time was not yet ripe for adopting any particular reform scheme, so the matter was again referred back to the various governments. Interest in the subject seems to be growing, and it may be that in the course of the next few years public opinion will crystallize in favor of some plan or type of plan of revision.

SOME MISTAKES I MADE IN TEACHING HIGH-SCHOOL MATHEMATICS

BY MARSHALL M. ABERNATHY
Centenary College, Shreveport, La.

Two years of teaching mathematics to college freshmen have given me ample opportunity to study at first hand the oft-discussed problem of the gap between high school and college work in this field of thought. I believe that the students whom I have taught during this time have been, on the whole, pupils of about the same caliber as those found at most of our colleges of today. Hence, I make bold to speak about a few of my impressions concerning them.

Working with this group has caused me to think back repeatedly to the time when I, as a high-school teacher, was engaged in the business of preparing students for just such work as I am now meting out to these latest students of mine. This retrospection has given me, I believe, a somewhat different viewpoint of the work of the high-school teacher, and I think that if I had my work to do over, I should make some revisions. By way of apology, I will say that I am constantly having to watch the same things in college teaching.

After the young mathematics teacher has overcome the initial timidity and nervousness which come with the teaching of his first few classes, he is quite apt to look around about him and compare his duty with that of teachers in other subjects. He finds, then, that other teachers, let us say those of English, seem to be eternally snowed under a pile of themes, notebooks, exercises, etc. He is then disposed to say to himself that his task is much easier than theirs. He finds that he already knows the subject matter to be presented, and that those papers which he has to grade apparently are quite easy to evaluate, because of the nature of his subject. He sometimes even goes so far as to congratulate himself that this soft berth is his reward for having tackled and mastered mathematics in college, while so many of his fellow-students did not.

The conscientious teacher, however, soon passes this stage. He finds, day by day, that there is more responsibility to be assumed in the teaching of mathematics than he realized during his first week or so. In fact, many teachers tell us that the longer they teach, the more difficult it becomes to satisfy themselves that the job is being done correctly.

In the beginning, however, it is quite natural that the young teacher should overlook some of the deeper aspects of his task. In my mind, there are six changes which I should make in my teaching if I had the task to do again—six aspects of the problem to which I would give greater emphasis. Having done this, I should consider my pupils a little better prepared, not only for college mathematics, but for the more perplexing problems of life as well.

The first of these would be to make a more determined effort to arouse the interest and enthusiasm of my pupils in the subject. It is truer of mathematics, I believe, than of any other subject, that to the stranger it is a lifeless, dry, and dull drudgery, while to the friend it is the very essence of charm and reasonableness. Realizing this now more than ever before, I should spare no pains to make friends for mathematics. Those who, for various reasons, come to high school with a preconceived dislike for mathematics, or with the belief that they "just can't do it," offer a very great challenge to the ingenuity, skill, and wisdom of the mathematics teacher. To let them slip on past without making an honest effort to change these false attitudes is to neglect an important opportunity. That so many come into, and pass through, college classes in this state of mind is, it seems to me, a reflection on our methods in both fields of teaching.

In the second place, I should make a more consistent attempt to create a willingness, even an eagerness, to stand by until the task is finished. No matter how great an interest a student may have in mathematics, unless he is willing to persevere, he cannot hope to succeed with it. Since the same thing is true of life, in all its activities, we

have here touched one of the chief sources of power in the study of mathematics. He who forms habits of persistence in any study, be it mathematics or not, cannot leave these habits at the schoolroom door when he graduates from school and enters the battle of life. Then, we cannot fail to do everything within our power to develop such habits. We are training citizens as well as mathematicians, and to succeed as either calls for constancy .

A third point of emphasis which a young teacher is likely to overlook is that no amount of facility in the juggling of the symbols and equations of mathematics can take the place of the true understanding which marks a mastery of mathematics. I am afraid that I, like many other young teachers, was too easily satisfied with dexterity, rather than demanding logical preciseness.

We teachers of mathematics like to say that mathematics is the "science of necessary conclusions," that "it teaches one to think in a more orderly manner," etc., etc., but we must face the sad fact that the world is full of people who have been exposed for the stipulated period to the beneficial influences of mathematics, but who perpetually fail to demonstrate by their actions that mathematics is so beneficial after all. It is quite possible that this is true because in the past we have failed to demand of our students that they do any great amount of careful thinking while they have been under our tutelage.

Professor Rankin, of Duke University, has sounded the same note in an article of his in *The Mathematics Teacher* for April, 1929:

> Our textbooks and those of us who teach mathematics are so intent that our students shall acquire a certain amount of technique in manipulating the symbols used in mathematics that we have bounded mathematics on the north by x, on the east by sin A, on the south by log x, and on the west by $\sqrt{-1}$ in about the same manner we learned to bound Ohio when we studied geography. The beauty and power of mathematics to set forth truth is lost sight of.

A fourth improvement which I should attempt in my teaching of mathematics would be to instill into my students a respect for scholarship in general, and for mathematical scholarship in particular. To this end I would point out to them from time to time the applications which mathematics has in other fields of learning, such as economics, chemistry, and various other sciences. I would indicate to them that no one can tell but that the seemingly most impractical theory may some day be the basis of scientific practice that will revolutionize some field of industry, as has frequently been the case in the past. I would indicate to them that generalization almost always precedes specific practice. I would, in short, endeavor to impress upon them the fact that scholarship is the coördinator and guiding genius of progress.

My fifth alteration would be to devote more time of my own to considering ways of securing maximum results as regards certain moral and spiritual benefits connected with the study of mathematics. For example, mathematics should train one to withhold judgment until all the facts are in; and it should help one to overcome prejudices, conceits, and emotion in the consideration of facts. What a need in the world for such thinkers! We constantly allow our preconceived notions of things to color our verdicts. Such is not the case in pure mathematical reasoning, however. Of what other subject in our curriculum is this so true? Where, then, can we better attempt to promote this disinterested method of weighing evidence? Again, even in mathematics we can do our bit to bring forth a citizenry enlightened as well as armored.

Further, there is a permanence and security about mathematical truth that is inspiring and often consoling. Truth in this field does not change from generation to generation. The theorems of Euclid are a lasting memorial to this fact. Is it impossible that the acquaintance with such lasting principles may be salutory on the mind of some youth confused with the multiplicity of customs, creeds, and faiths of our day? To find one thing that does not

change may be a great comfort. I should certainly not overlook this probability again.

In the sixth place, I should do all in my power to develop in my pupils, and in myself, that which is the very essence of all progress—initiative. How many pupils have you and I met who could do the task at hand easily enough "if you'll give me a hint"? On the other hand, have we not met just as many others in the world outside who always wait for the other man to get things started? Is there not some cause for the latter in the way we may dispose of the former? I should seek diligently for any conceivable method of encouraging that boldness backed by knowledge and confidence, which we call initiative.

At the same time, I should not be quite so prone as of yore to accept without careful scrutiny the advice of others as to how I should go about teaching my subject. I should, rather, experiment rather boldly and rather extensively to find the method best suited to my personality, and, more important still, to the natures and needs of my pupils.

"LESSONS IN MATHEMATICS"

BY C. E. CASTAÑEDA
Latin-American Librarian
The University of Texas

Under this title there appeared in Mexico an interesting little book in 1769 containing the lessons given by Señor Josef Ignacio Bartolache on this subject in the Royal University of Mexico. My curiosity aroused by the title, I took a look at the quaint booklet and was much surprised to find a brief, concise, and delightful exposition of mathematical principles presented with the strictest logic. "Its object," it declares, "is to instruct youth in the sciences most essential to society, such as mathematics." That the author had a high conception of the importance of this branch of human knowledge is evident from the following citation:

> As to its utility, it is well known that the greatest inventions, those which have been of greatest service to statesmen on land and sea, such as the development of commerce, the direction and command of military units, the attack and defense of fortified towns, the construction of beautiful buildings, all these have reached a state of perfection possible only after the cultivation of the mathematical sciences.

The author proceeds to enlarge upon the blessings enjoyed by the world as a result of the cultivation and development of mathematical knowledge. He points out the indebtedness of physics and medicine, of mechanics, of geographic discovery, and of astronomy. The purpose of the "Lessons," modestly declares the author, will be to explain the "mathematical method," defining the terms necessary for a beginner and setting down two propositions of the greatest importance. He promises to follow this introductory treatise with a complete arithmetic and a geometry, "the basis of all the other mathematical sciences." These in turn were to be followed by a book on mechanics, one on engineering, and others.

No evidence that the author published the subsequent treatises has been found, but the little book we have gives ample proof of the high state which the teaching of mathematics had reached in Mexico by the second half of the eighteenth century. It is quite plain that enthusiasm for the strict mathematical method of approach to the sciences was prevalent and that the advocates of inductive and deductive reasoning were beginning to hail the value of mathematics as a discipline in logical thinking. After giving many definitions of the various terms, he sets down the following proposition, which he proves to his satisfaction: "If all the natural sciences were treated by the mathematical method they would all be mathematical sciences."

The development of interest in scientific studies in Mexico during the last half of the eighteenth century was remarkable. A number of scientific journals were published, such as *El Mercurio Violante* in 1772, with "important and curious notes on various subjects of physics and medicine," and *Asuntos Varios Sobre Ciencias y Artes*, published in the same year and containing many interesting articles on science in general and mathematics in particular. Of no little interest is the space given in these scientific publications to the work of Benjamin Franklin in the field of electricity. Not only do we find frequent references to the genial diplomat of American Revolutionary fame, but a detailed funeral notice and eulogy of this unpretentious inventor.

A word about the life of Josef Ignacio Bartolache, professor of mathematics in the Royal University of Mexico, distinguished physician of his day, and a pioneer scientist, reveals better than anything else the interest in the study of the exact sciences in Mexico at this time. Born in Guanajuato, the richest mining district in Mexico, Bartolache was so poor that had not an influential gentleman of Mexico City taken an interest in him he would never have received a complete education. Brought to the capital of

the viceroyalty by his benefactor, he entered the famous old College of San Ildefonso to study philosophy, where scholasticism was still enthroned. After he obtained his master's degree, he studied theology; troubled by serious doubts, he began reading the philosophers of the French Revolution and soon found himself expelled from school for his liberal tendencies, his inquisitive mind, and his love of science. He took up the study of medicine and after five years of intensive study obtained a degree. While engaged in the study of medicine he became interested in mathematics, and this science captivated his whole mind by its rigid logic. By a queer coincidence he found himself teaching mathematics in the university shortly afterwards, as a substitute for his former teacher and friend.

The new instructor soon attracted attention by his new methods and liberal ideas and his class became crowded. This aroused envy, and before long he found himself without a job again. His mathematical mind did not permit him to turn permanently to medicine for a livelihood. Physical ailments could not be reduced to the certainty of a mathematical problem and solved without serious doubts. He at last made up his mind to abandon the practice of medicine and secure a position in the accounting department, where he labored faithfully during the remainder of his life. So efficient was he in the new position that he was soon put in charge of his division. His honesty and ability as an accountant are evidenced by the fact that after 11 years of service no error was ever discovered that reflected upon his character or his accuracy.

While attending to the exacting duties of his office, Bartolache found time to indulge in his favorite pastime, the study of mathematics and of scientific developments. He contributed to the various scientific journals and was a member of several commissions that fostered the scientific movement in Mexico. His contemporary, Antonio Alzate y Ramirez, after whom the most distinguished Mexican scientific society of today is named, and who had several heated disputes with Bartolache, generously paid him the highest

tribute by declaring that his erudition was vast, his judgment sound, his love of science and truth boundless. Such was the author of the "Lessons in Mathematics," a truly remarkable exposition of mathematical principles published before American independence.

THE MEANINGS OF NUMBERS

BY S. B. RED

Thorndike in his *Psychology of Arithmetic* states that one of the primary tasks of the elementary school is to teach the meanings of numbers. As simple as this may seem, it is something not generally understood even by high-school graduates. The charge is often made by college and university professors that freshmen do not have "number sense," and without question the charge is true. This phase of elementary mathematics is so important that a few suggestions on the subject should be in order.

The first meaning of integral numbers learned by the young pupil is the series meaning. This simply means that the child is able to call the numbers in their proper order, or in other words, to count by ones. This also implies that a child knows that 7 is one more than 6 and one less than 8, and so for all the other numbers. An understanding of this meaning is, of course, fundamental for all number work.

A second meaning of number is spoken of as the collection meaning. In this case a child associates the number 5 with 5 objects, 8 with 8 objects, etc. Certainly this is a fundamental meaning, and it should be said here that this as well as the series meaning must be well understood by those who make any progress at all in arithmetic. The next important meaning, and one not so well developed, is the ratio meaning. In this case a child should know that 4 is twice whatever stands for 2, or four times whatever stands for 1, etc. If — is 3, then —— is about 6. This meaning is important and no doubt should receive more attention than has been given it by the majority of teachers.

Next we have the relational meaning of number. Here the child should know that 6 is 2×3, $\frac{1}{2}$ of 12, $24 \div 4$, and so on for all of the other number facts. It can be seen here that there are a vast number of combinations which must be taught in order to develop properly this meaning.

Osburn, in his *Corrective Arithmetic*, lists over 1,600 fundamental number facts to be taught, and the pupil should be drilled on them until the correct response can be given automatically upon sight of the combination.

In order then to give the meaning of simple numbers it is necessary to cover the four phases mentioned. All of them are important, and should be taught thoroughly. All teachers recognize the first two mentioned, but as has been said before, there are few teachers who give enough attention to the ratio meaning. The last mentioned one is recognized by most teachers, but it is as a rule poorly done. This is not the teacher's fault in every case, but the teacher can do more than anyone else to remedy the situation. It should be remembered that if a child fails to learn these number facts in the period of school work in which he is supposed to learn them, the chances are that he will never know them. If he is allowed to form bad habits in this stage of learning, the habits will most likely follow him throughout his life. It should be remembered that it is much easier to form a correct habit in the beginning than it is to replace a bad habit with the correct one. To those who wish to read further on this phase of the teaching of arithmetic, Thorndike's *Psychology of Arithmetic* is recommended.

The next number work that is given involves fractions. It is strange to say, but true nevertheless, that the young pupil in working with fractions rarely thinks of them as numbers in the sense that he thinks of integers as numbers, and, of course, there are some differences, but it should be impressed upon the child that fractions are numbers and that they obey the same rules that are used in working with integers. By far the greater part of the trouble experienced by a child with fractions is due to his lack of understanding of the concept "fraction." Once this is understood, it is as easy to add similar fractions as it is to add similar objects. The problem $\frac{3}{7} + \frac{2}{7}$ should be as easy for a child as the problem 3 apples + 2 apples, and it will

be as easy if the child knows what a fraction is. The teacher then should stress the fact, in the work with fractions, that they are numbers just as integers are numbers, and in combining them we make use of the same principles that we use in the work with integers. The only new thing to be taught here is that both the numerator and denominator of a fraction may be multiplied or divided by the same number without altering the value of the fraction. This, of course, is fundamental in addition and subtraction and can be used to considerable advantage in division. Consider the division $\frac{\frac{a}{b}}{\frac{c}{d}}$. Multiplication of both numerator and denominator of this fraction by bd gives $\frac{ad}{bc}$. This is considered by some teachers as superior to the method usually given, that of inverting the terms of the divisor and multiplying. It makes use of a principle already mastered and, other things being equal, such a method should be given preference. In addition to being based on principles previously learned, the above method has the advantage of enlarging and clarifying the concept "fraction."

This part of arithmetic should then be thought of simply as a continuation and enlargement of the principles already learned and not as something new and unrelated to the child's experience. To be sure, some new concepts and ideas will have to be used, but they should be held at a minimum. In the words of Thorndike, "other things being equal, do not form two or three bonds when one will serve."[1]

The writer feels that it is unnecessary to discuss directed numbers in this article, since a very able discussion has appeared in a previous number of this *Bulletin*. The article was written by Professor J. W. Calhoun of The University of Texas and was published in the May, 1930, number.

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[1]Thorndike, E. L., *The Psychology of Arithmetic*, p. 101.

Something should be said concerning irrational numbers. It is a rare thing to find a high-school graduate who has the slightest idea of the nature of irrational numbers. They are spoken of as surds in a large number of books. One well-known text offers the following explanation:

> If n is a positive integer and a is a positive rational number which is not a perfect n^{th} power, then the $\sqrt[n]{a}$ is called a surd of the n^{th} order.
>
> Thus $\sqrt{6}$ is a surd of the second order,
> $\sqrt[3]{2}$ is a surd of the third order,
> $\sqrt{1.44}$ is not a surd, for $(1.2)^2 = 1.44$.

The author then shows by a few examples how surds are simplified, and follows this explanation with a list of exercises to be worked by the student.

It is not the purpose here to criticize adversely any particular textbook on algebra, but the writer feels certain that the explanations usually given leave the student with no definite idea as to the nature of irrational numbers. It should be said that no one advocates a course in the theory of numbers for high-school pupils, but some of the more interesting and elementary facts can be given. It is believed by many that high-school students would be highly interested to know that any integer divided by another integer will give as a quotient a repeating decimal. Consider the fraction $\frac{22}{7}$; division shows that $\frac{22}{7} = 3.142857142857\ldots$. Here the pattern 142857 repeats itself, so we can write this number to any number of decimal places without further division by merely repeating this pattern. Consider the fraction $\frac{3}{23}$; division gives $\frac{3}{23} = .130434782608695652173 9\ldots$. To divide further is unnecessary, because this pattern repeats itself. By a few such examples and a few words of explanation on the part of the teacher the student can be made to see that any integer divided by another integer

gives a repeating decimal. If the division ends, as in the case of $\frac{3}{4}$, we still have a repeating decimal in that the zeros are repeated. This idea that every fraction is a repeating decimal and every repeating decimal is a fraction should be understood by all, because it is to be used in explaining an interesting thing about irrational numbers. Consider again the fraction $\frac{22}{7}$. This is a very close approximation to π, but we have seen that it is a repeating decimal. The number π has the property that there is no pattern that repeats, and the high-school pupil can be told that it has been shown by higher mathematics that this is the way in which irrational numbers differ from rational numbers. If the student understands what is meant by incommensurable quantities (though this is rarely the case), he might be told that an irrational number is always obtained when two such quantities are divided. For example, the diagonal of a square divided by the side of the square gives $\sqrt{2}$, an irrational number.

It is easy to prove that the square root of 2 is not a repeating decimal: that is, one integer divided by another. The proof follows.

First, let us prove that if $\frac{a}{b}$ is an irreducible fraction, then $\frac{a^2}{b^2}$ is also irreducible. Every number may be resolved into prime factors, and obviously is equal to the product of these factors. Now the square of any number will be the product of all prime factors of the number, each taken twice. It can be seen then that the square of a number cannot contain any prime factor which the number does not contain.

Now if $\frac{a}{b}$ is an irreducible fraction, a and b can have no common factors; therefore, from what has been said above,

it follows that a^2 and b^2 can have no common factors, that is, $\frac{a^2}{b^2}$ is an irreducible fraction.

Now assume that $\sqrt{2} = \frac{a}{b}$, an irreducible fraction. Squaring, we get $2 = \frac{a^2}{b^2}$. This last statement is false, because a whole number cannot be equal to an irreducible fraction. It follows then that $\sqrt{2}$ cannot be a fraction, or in other words, a repeating decimal.

It is easy to show that logarithms are in general irrational numbers. Consider the logarithm of 9. If it can be shown that this logarithm is not a fraction (one integer divided by another), it follows that it is an irrational number. The indirect proof is offered here.

Suppose that $\log_{10} 9 = \frac{p}{q}$

Then $10^{p/q} = 9$

By raising both sides of this equation to the q^{th} power we get $10^p = 9^q$

But 10 to an integral power cannot be equal to 9 to an integral power; because 10 to any such power will end in zero and 9 to any integral power will never end in zero. Therefore, the logarithm of 9 cannot be a fraction, hence it is irrational.

Just how useful this concept is for high-school pupils few people know, because it has not been tried by a sufficient number of teachers to determine its true value, but it is believed by some that it is worth while in that it gives opportunity to enlarge and clarify the meanings of numbers.

INTERPOLATION

BY RICHARD GONZALEZ
San Antonio, Texas

The idea of correspondence plays an important role in mathematics. Relations such as

$x = 1, 2, 3, 4, \ldots\ldots\ldots$ $\quad x = 1, 2, 3, 4, \ldots\ldots\ldots$
$2x = 2, 4, 6, 8, \ldots\ldots\ldots$ $\quad x^2 = 1, 4, 9, 16, \ldots\ldots\ldots$

are simple correspondences which are taken up early in the study of arithmetic. One of the two variables between which correspondence holds is called the argument and the other is called the function of that argument. We will let a function y of an argument x be defined by the equation $y = f(x)$. To determine the value of y corresponding to an assigned value of x is not always easy; indeed, this is sometimes a lengthy and complex process. In such cases, we often make use of a table which gives the values of the function of the argument corresponding to certain selected values of the argument. One of the chief functions of interpolation is to assist us in using such a table.

For example, suppose we wish to determine the value of log 3.0125 from the following entries in a table of logarithms:

x	log x
3.00	0.477121
3.01	0.478567
3.02	0.480007

Interpolation, which may be described as "reading between the lines of a mathematical table," enables us to determine log 3.0125 very readily. The number 3.0125 is one-fourth of the way from 3.01 to 3.02. The difference between log 3.01 and log 3.02 is .001440, and one-fourth of this difference is 0.000360. Hence, the number one-fourth of the way from 0.478567 (log 3.01) to 0.480007 (log 3.02) is 0.478927. This corresponds to the number one-fourth of the way from 3.01 to 3.02; that is, log 3.0125 = 0.478927.

The above example illustrates a simple method of interpolation frequently used in mathematics. The general principle of this type of interpolation may be seen more readily with the help of the following table.

x			y = f(x)		
a	d_1	d	A	D_1	D
w			W		
b			B		

The values a, b, A = f(a), and B = f(b) are found in the table, and we wish to find W = f(w) corresponding to the value w which is between a and b. We determine the value of W by using the fact that W is the same part of the way from A to B as w is from a to b. The principle may be stated symbolically in the form $\frac{D_1}{D} = \frac{d_1}{d}$, where the symbols represent the distances indicated in the table above. Since D, d, and d_1 can be found from the entries in the table and the given value of w, $D_1 = D \times \frac{d_1}{d}$ can be computed, and thus the value of W is determined, for $W = A + D_1$.

It should be noted that interpolation will give an approximate value for W, and not an exact value, unless y is a linear function of x. If $y = mx + b$, however, the value of y corresponding to a value of x can be determined exactly by interpolation, as the following figures indicate:

x			y = 2x + 1			
6	d_1	d	13	D_1	D	$D_1 = D \times \frac{d_1}{d} = 2\frac{.5}{1}$
w = 6.5			W = ?			
7			15			$D_1 = 1$

By interpolation, we determine the value of W to be $13 + 2\frac{.5}{1}$, or W = 14. If we substitute the value 6.5 for x

in the equation $y = 2x + 1$, we find that $W = 2(6.5) + 1 = 14$.

In case y is not a linear function of x, the value of W determined by the method of interpolation will not be exact, but it will be a fairly close approximation to the correct answer. We will take $y = x^2$ to illustrate this fact.

x	$y = x^2$
1	1
2	4
$w = 2.67$	$W = ?$
3	9
4	16

The difference between 3^2 and 2^2 is 5, and $0.67 \times 5 = 3.35$. Therefore, $W = 4 + 3.35 = 7.35$. We can check this answer by actually squaring w: $2.67 \times 2.67 = 7.1289$. It is clear that interpolation does not yield an exact answer in this case. We can also illustrate this by computing 3^2 by means of interpolation. Since 3 is one-half of the way between 2 and 4, this method of interpolation would give 10 (the number one-half of the way from 2^2 to 4^2) as the value of 3^2. This answer is obviously only an approximation.

If $y = x^2$, it is a fairly simple matter to compute the value of y directly from a given value of x. If $y = \log x$, however, the easiest method of obtaining an approximate answer is by interpolation in a table. The logarithm of a given number can be computed without particular difficulty by means of an infinite series, but such a method is too long to be employed in the ordinary use of logarithms. Interpolation is a particularly valuable device for determining the value of a function of an argument in those cases where it is difficult to evaluate the function by simple arithmetical operations on x.

The simple method of interpolation discussed above assumes that within a given interval the graph of the function is a straight line. This assumption is correct when y

is a linear function of x, and in other cases it is a reasonable assumption for relatively small intervals. Interpolation can be extended to cases where the values of a function are assumed to satisfy a relation of the form $y = a + bx + cx^2 + \ldots$, which is more general than the relation $y = a + bx$. This extension involves the use of second, third, and higher differences.

Inverse interpolation is the process of finding the value of the argument corresponding to a given value of the function intermediate between two tabulated values of the function.

x	$y = x^2$
1	1
2	4
$w = ?$	$W = 7.7$
3	9

The difference between 3^2 and 2^2 is 5, and $W - 4 = 3.7$, so $w = 2 + (1)\frac{3.7}{5} = 2.74$. If we square 2.74, we obtain the number 7.5076, which is less than W by 0.1924. If we compute the square of 2.74 by interpolation in the table, however, we get $W = 7.7$.

The notion of interpolation is quite frequently used, and in its simpler form it presents no difficulties. Once the principle of interpolation is properly grasped, there is no need for any rule concerning the process of reading between the lines of a table. In spite of its simplicity, interpolation is a very useful mathematical device, and it often makes the solution of a problem much easier than would otherwise be possible.